Aggregate return on investment and investment decisions: a new perspective

Aggregate Return On Investment and investment decisions: A cash-flow perspective

Carlo Alberto Magni

Department of Economics, University of Modena and Reggio Emilia
CEFIN– Center for Research in Banking and Finance

Abstract. The recent notion of Average Internal Rate of Return (AIRR) [Magni 2010, *The Engineering Economist*, 55(2), 150-180] completely solves the long-standing problem of the internal rate of return (IRR). While the AIRR is a *return* measure, this paper presents a *cash-flow* measure, namely the ratio of net cash flow (i.e., cash inflows minus cash outflows) to capital invested, which we call Aggregate Return On Investment (AROI). It is a purely internal measure because, unlike the AIRR, it does not depend on the market rate, and is a *return* measure, for it is a mean of one-period return rates, weighed by the outstanding capitals. The AROI is reliable in both accept/reject decisions and project ranking, in association with an appropriate hurdle rate, economically significant: the *comprehensive* cost of capital (CCOC), which takes into account not only the interest foregone on the capital actually employed, but also the interest foregone on the capital that is given up by the investor. This perspective enables one to decompose the project NPV into an excess-rate share and an excess-capital share. The traditional IRR is just a particular case of both AIRR and AROI, but the latter approach has the advantage that the IRR's nature (rate of return versus rate of cost) does not depend on the market rate and is unambiguously determined by the capital invested.

Keywords. Decision analysis, investment criteria, rate of return, cash flow, comprehensive cost of capital.

Introduction

The Internal Rate of Return (IRR) has a respectable ancestry, being conceived in 1935-36 by such economists as Boulding (1935, 1936) and Keynes (1936). While massively used by scholars, managers, analysts, professionals, practitioners, policy regulators, its shortcomings are well known, among which possible inexistence and incompatibility with Net Present Value (NPV). Recently, some interesting contributions have appeared in the literature. Among others, Hazen (2003) has solved the problem of multiple real-valued IRRs by linking the present value of the outstanding capitals of a project with the difference between any internal rate of return and the cost of capital. Osborne (2010) has derived the NPV of a project as a function of all (real-valued and complex-valued) roots of the IRR equation. The meaningfulness of complex-valued roots is corroborated by the analysis of Pierru (2010). Magni (2010a) has introduced a profitability index which completely solves the IRR problems with no need of making recourse to complex-valued numbers; the author presents the Average Internal Rate of Return (AIRR), which is a *return* measure, obtained as a mean of one-period internal rates of return, which are weighed by the present values of the outstanding capitals of the project. The AIRR supplies the correct solution in both accept/reject decisions and project ranking. As the weights of the mean are present values, and as the present values depend on the market rate, the AIRR itself depends on the market rate. This implies that, for a fixed stream of capitals, the AIRR may change under changes in the market rate and the very financial nature of the project may change, turning from investment to borrowing or vice versa.

This paper focuses on an intuitive *cash-flow* measure: the ratio of the project's net cash flow (i.e., cash inflows minus cash outflows) to the aggregate capital employed. We call this ratio Aggregate Return On Investment (AROI). Such a measure is *purely internal* because it is independent of the market rate and is a *return* measure: it may be obtained as a mean of the project period return rates weighed by the outstanding capitals. The AROI may be soundly used for accept/reject decisions as long as it is contrasted with a hurdle rate which takes account of the capital foregone by the investor: such a hurdle rate is a *comprehensive* cost of capital (CCOC). Also, maximization of residual rate of return (i.e., AROI minus CCOC) generates the same ranking as the NPV ranking.

The paper is structured as follows. Section 1 presents the mathematical notation and some preliminary notions. Section 2 briefly summarizes the AIRR criterion presented in Magni (2010a). Section 3 introduces the notion of comprehensive cost of capital (CCOC) and presents the Aggregate Return On Investment (AROI) as a particular case of AIRR where market rate is set to zero. It is shown that, irrespective of the market rate, the AROI is reliable in accept/reject decisions, as long as the CCOC is used as a hurdle rate. Section 4 shows that the AROI, introduced as a *return* measure, is actually a *cash-flow* measure: cash inflows minus cash outflows divided by capital. Section 5 shows that the market rate may be used as a hurdle rate as well, provided that a well-specified market-determined capital stream is selected for describing the project. Section 6 shows that the IRR is just a particular case of the AROI and that the financial nature of an IRR (rate of return or rate of cost) does not change under changes in the market rate. Section 7 deals with project ranking: it shows that maximization of the residual rate of return (AROI minus CCOC) is equivalent to NPV maximization, provided that equivalent capital streams are chosen for all the projects involved in the ranking. It is also shown that choices between two mutually exclusive projects may be accomplished by the application of the acceptability criterion to the incremental cash flow stream of a project over the other. Section 8 summarizes the differences between the AIRR criterion and the AROI criterion and shows that the AROI approach enables one to decompose the NPV into two shares: a rate margin and a capital margin. A straightforward generalization to variable market rates is also illustrated. Some concluding remarks end the paper.

1. Mathematical notation and some preliminary notions

A *project* or *cash flow stream* is a sequence $\boldsymbol{x} = (x_0, x_1, x_2, \dots, x_T) \in \mathbb{R}^{T+1}$ of cash flows. The net present value (NPV) of project $\boldsymbol{x}$ is

$$PV(\boldsymbol{x} \mid r) := \sum_{t=0}^{T} x_t \cdot (1+r)^{-t}$$

where $r > -1$ is the market rate. The net future value, denoted by $PV_t(\boldsymbol{x} \mid r) := PV(\boldsymbol{x} \mid r)(1+r)^t$, $t \geq 1$, is the future value of $PV(\boldsymbol{x} \mid r)$ at some future date. If $t = T$, the net future value is called Net Final Value (NFV). A project is profitable (or acceptable or worth undertaking) if and only if $PV(\boldsymbol{x} \mid r) > 0$. An internal rate of return for project x is a constant rate $k \neq -1$

such that $PV(\boldsymbol{x} \mid k) = 0$ or, which is the same, $PV_T(\boldsymbol{x} \mid k) = \sum_0^T x_t \cdot (1+k)^{T-t} = 0$. Let $c_t \in \mathbb{R}$, $t = 0,1,\dots,T$, and let

$$R_t \coloneqq c_t + x_t - c_{t-1}, \qquad t = 1,\dots,T, \qquad \text{with} \quad c_0 = -x_0, \qquad c_T = 0. \tag{1a}$$

The term c_{t-1}, which we call *outstanding capital* (see also Lohmann 1988), represents the capital invested (or borrowed) in the period $[t-1, t]$, and the amount $(c_t + x_t)$ is the end-of-period payoff, so R_t is the *return* (interest) generated by the project in that period. If $c_{t-1} \neq 0$, (1a) may be written as

$$c_t = c_{t-1}(1 + k_t) - x_t, \qquad t = 1,\dots,T, \qquad \text{with} \quad c_0 = -x_0, \qquad c_T = 0 \tag{1b}$$

where $k_t := R_t / c_{t-1}$ is a period rate of return. Equation (1b) says that the capital invested in the project at the beginning of a period increases by the return generated in the period but decreases (or increases) by the amount x_t, which is received (or paid) by the investor. Any vector $\boldsymbol{c} = (c_0, c_1, \dots, c_{T-1}) \in \mathbb{R}^T$ satisfying (1) is called an *investment stream* or *capital stream*.[1] If the outstanding capital c_{t-1} is the book value of the project, the period rate k_t represents a book (i.e., accounting) rate of return (for example, the well-known Return On Investment). The capital stream such that $k_t = r$ for all t will be denoted by $\boldsymbol{c}^* = (c_0^*, c_1^*, \dots, c_{T-1}^*)$, where $c_0^* = c_0$.[2] Equations (1a)-(1b) may be framed as

$$-c_{t-1} + \frac{c_t + x_t}{1 + k_t} = 0 \tag{2a}$$

so the period rate is indeed a classical *internal* rate of return. Rewriting (2a) as

$$c_{t-1}(1 + k_t) = c_t + x_t \tag{2b}$$

one may accept $k_t = -1$. It is easy to see that such an equation leads to $c_T = -\sum_{t=0}^T x_t(1 + k_{t+1})(1 + k_{t+2}) \cdots (1 + k_T)$. Letting $PV_T(\boldsymbol{x}, \boldsymbol{k}) := \sum_{t=0}^T x_t(1 + k_{t+1})(1 + k_{t+2}) \cdots (1 + k_T)$ and using the terminal condition $c_T = 0$, one gets to $PV_T(\boldsymbol{x}, \boldsymbol{k}) = 0$, which means that the sequence $\boldsymbol{k} = (k_1, k_2, \dots, k_T) \in \mathbb{R}^T$ of one-period IRRs represents an *internal return vector* (see Weingartner 1966; Peasnell 1982; Peccati 1989; Magni 2009). There are infinite sequences $\boldsymbol{k}$ of real-valued numbers that satisfy $PV_T(\boldsymbol{x}, \boldsymbol{k}) = 0$; an IRR (if it exists in the real interval) is only

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[1] Hazen (2003, 2009) and Magni (2010) uses the expression "investment stream". In this paper we prefer to use the expression "capital stream", to avoid any misunderstanding (it is not a stream of outlays).

[2] The terminal value c_T^* of the capital stream $\boldsymbol{c}^*$ is univocally determined by cash flow stream and market rate, which means, in general, that $c_T^* \neq 0$.

a particular case of internal return vector such that all components are constant: $\boldsymbol{k} = (k, k, \dots, k)$.

Consider now the ratio

$$\bar{k}_r := \frac{\sum_1^T R_t (1+r)^{-t}}{\sum_1^T c_{t-1}(1+r)^{-t}}. \tag{3a}$$

If $c_t \neq 0$ for all $t < T$, eq. (3a) may be rewritten as a weighted average of period rates:

$$\bar{k}_r := \frac{\sum_1^T k_t c_{t-1}(1+r)^{-(t-1)}}{PV(\boldsymbol{c}, r)} = \frac{\sum_1^T k_t c_{t-1}(1+r)^{-(t-1)}}{\sum_1^T c_{t-1}(1+r)^{-(t-1)}}\,. \tag{3b}$$

The ratio above is labeled *Average Internal Rate of Return* (AIRR). If the market rate is zero, the AIRR boils down to

$$\bar{k}_0 := \frac{\sum_1^T R_t}{PV(\boldsymbol{c}, 0)} = \frac{\sum_1^T R_t}{\sum_1^T c_{t-1}}. \tag{3c}$$

Equation (3c) is aggregate return divided by aggregate capital, so we name it "Aggregate Return On Investment" (AROI). One may also write

$$\bar{k}_0 = \frac{\sum_1^T k_t c_{t-1}}{C} \tag{3d}$$

with $C := \sum_1^T c_{t-1}$, as long as $c_t \neq 0$ for all $t < T$, so the zero-rate AIRR is a mean of period rates weighed by the (plain vanilla) outstanding capitals.

Equations $(3a)$- $(3c)$ define *return* measures. A *cash-flow* measure is the ratio of net cash flow (cash inflows minus cash outflows) to the capital invested at time 0.

$$\text{Net Cash Flow on Capital} = \frac{x_0 + x_1 + \cdots + x_T}{c_0}. \tag{4}$$

Simple and intuitive as it is, it apparently cannot be used as a profitability index, as it disregards time (there is no discounting) and considers cash flows rather than returns. In fact, the next sections will show that this ratio is a reliable rate of return.

Lastly, let $\boldsymbol{c^1} = (c_0^1, c_1^1, \dots, c_{T-1}^1)$ be a capital stream such that $\sum_0^{T-1} c_t^1 = K$. The capital stream $\boldsymbol{c^2} = (c_0^2, c_1^2, \dots, c_{T-1}^2)$ is said to be *equivalent* to $\boldsymbol{c^1}$ if $\sum_0^{T-1} c_t^2 = K$. The set of all capital streams $\boldsymbol{c}$ such that $\sum_0^{T-1} c_t = K$ for some $K \in \mathbb{R}$ will be called *investment class*.

2. The AIRR and the NPV

Magni (2010a) dismisses the traditional IRR equation and makes use of the AIRR to solve the IRR problems. The AIRR has the same economic status as the NPV. This section summarizes his main results.

Theorem 2.1. (*Acceptability*). For any capital stream $\boldsymbol{c} \in \mathbb{R}^T$,

(i) if $PV(\boldsymbol{c} \mid r) > 0$, project $\boldsymbol{x}$ is acceptable if and only if $\bar{k}_r > r$

(ii) if $PV(\boldsymbol{c} \mid r) < 0$, project $\boldsymbol{x}$ is acceptable if and only if $\bar{k}_r < r$

(iii) project $\boldsymbol{x}$ is value-neutral (i.e. NPV = 0) if and only if $\bar{k}_r = r$

Note that in the above theorem an implicit definition of investment and borrowing is given:

If $PV(\boldsymbol{c} \mid r) > 0$ then the project is a *net investment*

If $PV(\boldsymbol{c} \mid r) < 0$ then the project is a *net borrowing.*

This definition generalizes the definition given by Hazen (2003) for multiple IRRs.[3] If the project is a net investment, the AIRR is a rate of return (i.e., lending rate); if the project is a net borrowing, the AIRR is a rate of cost (i.e., borrowing rate). Theorem 2.1 enables one to compute a rate of return for any project and to correctly solve any accept/reject decision problem. Computationally, the steps are: (a) arbitrarily pick a capital stream $\boldsymbol{c}$ so that the project is described as either a net investment or a net borrowing, (b) compute the corresponding one-period return rates and their average, (c) if the project is a net investment (borrowing), accept the project if and only if the AIRR is greater (smaller) than the market rate.

A project may be described by the fundamental triplet

$$\left(PV(\boldsymbol{c}|r), \quad \bar{k}_r, \quad r\right)$$

which univocally determines the NPV: precisely,

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[3] This definition is essential in correctly interpreting the financial nature of the IRR. Until Hazen (2003) the notion of investment and borrowing was connected to the sign of the outstanding capital c_t (see Teichroew, Robichek and Montalbano 1965a,b); but this perspective brings about unfavorable cases where the sign of c_t changes over time, which implies that neither the IRR's financial nature (lending rate versus borrowing rate) nor the project's financial interpretation (investment versus borrowing) is univocal. Hazen's (2003) definition (and Magni's, 2010a, generalization) is important for it sweeps away such 'mixed' cases and gives the opportunity of establishing the exact nature of any project and any rate of return.

$$\frac{PV(\boldsymbol{c}|r)(\bar{k}_r - r)}{(1+r)} = \frac{PV_1(\boldsymbol{x} \mid r)}{(1+r)} = PV(\boldsymbol{x} \mid r) \tag{5}$$

The AIRR is then a hyperbolic function of $PV(\boldsymbol{c}|r)$:

$$\bar{k}_r(PV(\boldsymbol{c}|r)) = r + \frac{PV_1(\boldsymbol{x} \mid r)}{PV(\boldsymbol{c}|r)} \tag{6}$$

Thus, a unique AIRR corresponds to a determined $PV(\boldsymbol{c}|r)$.[4] Given that capital streams may be arbitrarily chosen by the analyst, if competing projects are to be ranked, one may pick, for all projects, present-value equivalent streams (i.e., giving rise to the same $PV(\boldsymbol{c}|r)$) so that, from (5), the following theorem is derived.

Theorem 2.2. (*Project ranking*)**.** Consider competing projects $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$**.** The ranking of the projects via their AIRRs is equal to the NPV ranking, provided that the capital streams of the projects are chosen so as to be present-value equivalent.

Among the infinite capital streams, consider the one derived by a real-valued IRR.

Theorem 2.3. (*IRR as AIRR*). A (real-valued) IRR is a particular case of AIRR generated by the set of those capital streams which are PV-equivalent to the IRR-implied capital stream.

3. Aggregate Return On Investment and Comprehensive Cost of Capital

If the investor rejects $\boldsymbol{x}$**,** an alternative course of action open to her is to replicate the investment in the market. That is, the investor invests $c_0 = -x_0$ in the market and periodically withdraws x_t (or injects x_t , if positive) at time *t*. Let c^*_{t-1} be the outstanding capital of this *market investment* as of time $t-1$ (beginning of the interval $[t-1, t]$). Then, vector $\boldsymbol{c}^* = (c^*_0, c^*_1, \dots, c^*_{T-1})$ is the market capital stream. The end-of-period value c^*_t is computed as

$$c^*_t = c^*_{t-1}(1+r) - x_t \qquad t = 1,2,\dots,T \qquad\qquad c^*_0 = -x_0. \tag{7}$$

This means that the capital invested in the market at the beginning of the period increases by the market rate r and decreases by the amount x_t (or increases if $x_t < 0$). The market capital c^*_{t-1} then expresses the capital that could be alternatively invested at time $t-1$ if the

[4] Note that $PV(\boldsymbol{c}|r)$ (not $\boldsymbol{c}$) is in a biunivocal correspondence with $\bar{k}_r$. Therefore, an AIRR does not change under changes in the capital stream, as long as $PV(\boldsymbol{c}|r)$ doe not change.

investor did not undertake the project. From a cash-flow perspective, if the market investment were undertaken, the investor would receive the same cash flows as the project, but she would be left, at time T, with the terminal value c_T^* of the market investment. In other words, denoting with $\boldsymbol{x}^*$ the market investment, if the investor accepts to receive $\boldsymbol{x} = (x_0, x_1, x_2, \dots, x_T)$ she renounces to receive the cash flow stream $\boldsymbol{x}^* = (x_0, x_1, x_2, \dots, x_T + c_T^*)$.

What is then the opportunity cost incurred by the investor? Consider the case where the investor undertakes $\boldsymbol{x}$. In each period, she

- invests c_{t-1} at the beginning of the period
- earns k_t on each unit of capital invested.

Hence, the project return is $R_t = k_t c_{t-1}$. Conversely, consider the alternative case where the investor undertakes the market investment. In each period, she

- invests c_{t-1}^* at the beginning of the period
- earns r on each of unit of capital invested.

Hence, the market-investment return is $R_t^* \coloneqq r c_{t-1}^*$. Such a return takes account that the capital invested in the market alternative is not c_{t-1}, but c_{t-1}^*. So, the market return R_t^* is a *comprehensive* opportunity cost, which may be viewed as the joint effect of two conceptually different components, related to a different capital invested and a different return rate applied to the capital. Precisely, the interest foregone by the investor may be split into

(i) rate component: interest foregone on the capital actually employed in the project

(ii) capital component: interest foregone on the excess capital invested in the market:

$$R_t^* = \overbrace{r c_{t-1}}^{\text{rate component}} + \overbrace{r(c_{t-1}^* - c_{t-1})}^{\text{capital component}} \tag{8}$$

The cost of capital in the interval $[t-1, t]$ is a multiple of the market rate:

$$\rho_t \coloneqq \frac{R_t^*}{c_{t-1}} = r\left(\frac{c_{t-1}^*}{c_{t-1}}\right),$$

where the term c_{t-1}^*/c_{t-1} is a correction factor which adjusts for the different capital invested. Given that the capital invested in the t-th period is c_{t-1} for $t = 1,2, \dots, T$, the sum of the capitals represents the grand total capital invested in the project. We denote by $C \coloneqq$

$\sum_1^T c_{t-1}$ this aggregate capital; likewise, we denote by $C^* := \sum_1^T c_{t-1}^*$ the aggregate *market* capital. Therefore, the overall cost of capital is

$$\bar{\rho} := r\left(\frac{C^*}{C}\right) \tag{9}$$

which is here named *comprehensive* cost of capital (CCOC). The ratio C^*/C expresses the proportion of the market investment's aggregate capital to the project's aggregate capital. For example, if $r = 10\%$ and $C^*/C = 1.2$, the investor is not merely renouncing to earn 10% on the capital C invested in the project; she is also renouncing to invest an extra 20% of capital at a 10% return! Therefore, the cost of capital is 10%(1.2)=12%. The excess 2% is just due to the additional interest foregone on the excess capital $(c_{t-1}^* - c_{t-1})$.[5]

If $c_{t-1} \neq 0$ for every $t = 1,2,\dots,T$, the rate $\bar{\rho}$ may be represented as a weighted average of the ρ_t's:

$$\bar{\rho} = \frac{\sum_1^T r c_{t-1}^*}{C} = \frac{\sum_1^T \rho_t c_{t-1}}{\sum_1^T c_{t-1}}.$$

We now prove that the NPV of $\boldsymbol{x}$ may be expressed in terms of difference between project returns and market-investment returns.

Lemma 3.1. Consider an arbitrary capital stream $\boldsymbol{c} = (c_0, c_1, c_2, \dots, c_{T-1})$. Then, the following equality holds:

$$PV(\boldsymbol{x} \mid r) = \sum_{t=1}^{T} (R_t - R_t^*) \cdot (1+r)^{-T}. \tag{10}$$

Proof: By definition of R_t and R_t^*, one gets

$$\sum_1^T (R_t - R_t^*) = \sum_1^T [(c_{t-1} - c_t + x_t) - (c_{t-1}^* - c_t^* + x_t)]$$
$$= c_0^* - c_0 + c_T - c_T^*.$$

Given that $c_0^* = c_0 = -x_0$ and $c_T = 0$, then $\sum_{t=1}^T (R_t - R_t^*) = -c_T^*$. Owing to (7), $c_T^* = -\sum_0^T x_t (1+r)^{T-t}$, whence the thesis is obtained.■

The term $(R_t - R_t^*)$ in (10) represents a "residual income", that is, it measures the return in excess of what could be earned by investing the capital c_{t-1}^* at the market rate r. If $c_{t-1} \neq 0$ for every $t = 1,2,\dots,T$, the rates k_t and ρ_t are well-defined so that (10) may be framed as

[5] If $0 < C^* < C$, the greater capital invested in the project tends to offset the rate component, so the CCOC turns out to be smaller than the market rate.

$PV(\boldsymbol{x} \mid r) = \sum_1^T c_{t-1}(k_t - \rho_t)(1+r)^{-T}$. The margin $(k_t - \rho_t)$ measures the residual income per unit of capital invested in the project.

Remark. It is worth noting that the residual income involved in (10) is different from the standard residual income found in the literature and employed by Lohmann (1988), Peccati (1989), Hazen (2003) and Magni (2010a). The standard residual income is well-rooted in managerial economics and accounting, and is made to depend on the actual capital invested, not on the market-investment capital: it is therefore computed as $(R_t - rc_{t-1})$ (see Peasnell 1982; Brief and Peasnell 1996). In a nutshell, we replace the standard opportunity cost $R_t = rc_{t-1}$ with the all-inclusive opportunity cost $R_t^* = \rho_t c_{t-1}$ (see Magni 2009, 2010b. For an axiomatic approach to residual income, see Ghiselli Ricci and Magni 2010).

Consider the value taken on by the AIRR when the market rate is zero: $\bar{k}_0 = \sum_1^T R_t \,/\, \sum_1^T c_{t-1}$. This rate of return does not include the market rate, so it is a *purely internal* rate of return. As anticipated, we label it Aggregate Return On Investment (AROI). We now show that project acceptability is easily established, whatever the market rate, if the AROI is contrasted with the CCOC as the hurdle rate.

Theorem 3.1. (*Acceptability*). For any capital stream $\boldsymbol{c} \in \mathbb{R}^T$,

(i) if $C > 0$, project $\boldsymbol{x}$ is acceptable if and only if $\bar{k}_0 > \bar{\rho}$

(ii) if $C < 0$, project $\boldsymbol{x}$ is acceptable if and only if $\bar{k}_0 < \bar{\rho}$

(iii) project $\boldsymbol{x}$ is value-neutral (i.e. NPV = 0) if and only if $\bar{k}_0 = \bar{\rho}$

Proof: Owing to Lemma 3.1 and using the definition of $\bar{\rho}$ and $\bar{k}_0$

$$PV(\boldsymbol{x} \mid r) = \sum\nolimits_1^T (R_t - rc_{t-1}^*) \cdot (1+r)^{-T} = \sum\nolimits_1^T R_t (1+r)^{-T} - r \sum\nolimits_1^T c_{t-1}^* \,(1+r)^{-T}$$

$$= \frac{C}{(1+r)^T} \cdot \left(\frac{\sum_1^T R_t}{C} - \frac{rC^*}{C} \right) = C \cdot \frac{(\bar{k}_0 - \bar{\rho})}{(1+r)^T} \tag{11}$$

which holds for any arbitrary capital stream (and for any market rate). Hence, the thesis follows immediately.■

Equation (11) is enlightening: it tells us that the project may be entirely described by the fundamental triplet

$$(C, \;\; \bar{k}_0, \;\; \bar{\rho})$$

which contains the fundamental financial variables affecting a project's profitability: capital invested, project rate of return, (comprehensive) cost of capital. Theorem 3.1 also supplies an implicit definition of investment and borrowing based on the capital invested:

- If $C > 0$ then the project is a *net investment*
- If $C < 0$ then the project is a *net borrowing*.

A project may always be viewed as either an investment or a borrowing of any monetary amount at the decision maker's discretion: the overall economic analysis does not change and the decision is univocal. The investor needs only select the preferred capital stream, check the nature of the project and compare the AROI with the CCOC.

For any fixed $K \in \mathbb{R}$, the equation $K = \sum_1^T c_{t-1}$ has infinite solutions, so there are infinitely many capital streams $\boldsymbol{c}$ which give rise to the same AROI. The same holds for the CCOC. In other terms, AROI and CCOC are *uniquely* associated with an investment class. Precisely, both the function relating AROI and C and the function relating CCOC and C are bijections:

$$\bar{k}_0(C) = \frac{\sum_1^T R_t}{C} \qquad\qquad \bar{\rho}(C) = \frac{\sum_1^T R_t^*}{C}.$$

Figure 1 illustrates the graphs of $\bar{k}_0(C)$ and $\bar{\rho}(C)$ for a positive-NPV project. The AROI is greater (smaller) than the comprehensive CCOC for every positive (negative) C. Graphically, $PV_T(\boldsymbol{x} \mid r)$ is the area of any rectangle with base $\overline{0C}$ and height $|\bar{k}_0(C) - \bar{\rho}(C)|$.

EXAMPLE. Consider the cash flow stream $\boldsymbol{x} = (-10, 30, -25)$ studied by Hazen (2003, p. 44) and Magni (2010a, p. 161), where a market rate equal to 10% is assumed. The project NPV is $PV(\boldsymbol{x} \mid 10\%) = -3.39$, so the project is not profitable. No real IRR exists. One may conveniently choose, at discretion, any capital stream and then compute the corresponding (real-valued) AROI. Table 1 reports some capital streams (the same used by Magni 2010a) and the associated AROIs and CCOCs. The answer is the same as that given by the NPV: the project should be rejected. For example, the third capital stream tells us that the project consists in borrowing 18 euros, so the AROI (interpreted as a rate of cost) is equal to 27.78%, whereas the CCOC is equal to $5\% = 10\% \cdot (\frac{9}{18})$. Hence, the project is not worth undertaking.

Note that $-18 \cdot (27.78\% - 5\%)(1.1)^{-2} = -3.39$ =NPV. The same result is obtained from the other patterns.

Remark. To contrast AROI and CCOC boils down to checking the sign of the residual rate of return (RRR $= \bar{k}_0 - \bar{\rho}$). If the project is described as a net investment, positive (negative) sign means value creation (destruction), if the project is described as a net borrowing, the reverse holds. As we will see, the RRR plays a major role in project ranking.

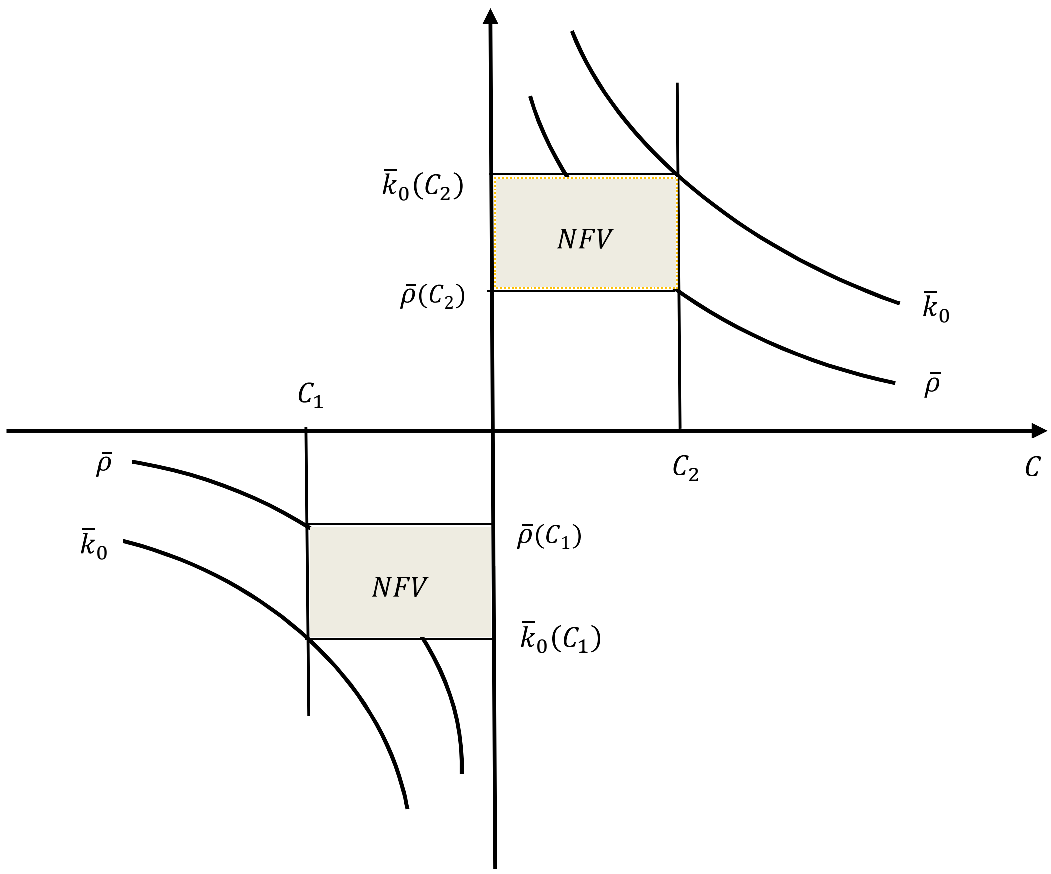


Figure 1. Any triplet $(C, \bar{k}_0(C), \bar{\rho}(C)$) univocally determines the NFV.

Table 1. Complex-valued IRRs, real-valued AROIs (market rate=10%)

Time		0	1	2	C, C^*	Type	AROI (%)	CCOC
Cash Flows		-10	30	-25				
NPV		-3.39						
	$\boldsymbol{c}^1$	10	-6	0	4, –9	investment	–125%	–22.5%
	$\boldsymbol{c}^2$	10	-20	0	–10, –9	borrowing	50%	9%
	$\boldsymbol{c}^3$	10	-28	0	–18, –9	borrowing	27.78%	5%
	$\boldsymbol{c}^4$	10	0	0	10, –9	investment	–50%	–9%

The following section shows that the AROI, introduced as a particular case of AIRR, is actually a *cash-flow* measure, for it is directly derived from cash flows: it is just the net cash flow per unit of capital invested.

4. AROI is a cash-flow measure

We have introduced the AROI as a return measure, describing it as a return-on-capital index. This section shows that the AROI is actually a *cash-flow* measure, so that the AROI approach represents a link between return measures and cash-flow measures.

Proposition 4.1. (*AROI as a cash-flow measure*)**.** The AROI is the net cash flow (i.e., cash inflows minus cash outflows) per unit of aggregate capital invested.

Proof: from $R_t = c_t - c_{t-1} + x_t$ one gets

$$\sum_{t=1}^{T} R_T = -c_0 + x_1 + x_2 + \cdots + x_T + c_T = \sum_{t=0}^{T} x_t. \tag{12}$$

Therefore,

$$\bar{k}_0 = \frac{x_0 + x_1 + \cdots + x_T}{C} \tag{13}$$

which is just the ratio of net cash flow to capital invested. ■

Particularly important is the case where the analyst is willing (or required) to provide a rate of return on the *initial* capital invested. This is a rather natural choice: after all if one invests

10 euros at time 0 in a T-period project, one naturally asks for the return on those 10 euros. Therefore, picking $C = c_0$ in (13), one finds that the initial-capital AROI is just the *Net Cash Flow on Capital* (*NCFC*) introduced in section 1:

$$NCFC = \bar{k}_0(c_0) = \frac{x_0 + x_1 + \cdots + x_T}{c_0}. \tag{14}$$

Thanks to Proposition 4.1, Theorem 3.1 implies the following corollary:

Corollary 4.1. (*Acceptability*)**.** Project $\boldsymbol{x}$ is acceptable if and only if $NCFC > \bar{\rho}(c_0)$ (sign is reversed if $c_0 < 0$) and value-neutral if and only if $NCFC = \bar{\rho}(c_0)$

Therefore, AROI is a cash-flow profitability index and *NCFC* is a particular case of AROI.

Remark. We have just shown that the AROI is directly obtained from the project's cash flows. Now, if cash flows are given, there is no need of calculating it indirectly from the project's period return rates: one just has to subtract cash outflows from cash inflows and divide by capital. Moreover, if *NCFC* is employed, the fixation of a capital stream is actually unnecessary: the computational steps reduce to

a) compute the net cash flow and divide it by initial capital ($= \sum_0^T x_t / c_0$)
b) compute the market-capital stream, and multiply the market rate by the correction factor ($= r \cdot C^*/c_0$)

Remark. The CCOC itself may be written in terms of cash flows: from $R_t^* = c_t^* - c_{t-1}^* + x_t$ one gets

$$\sum_{t=1}^{T} R_t^* = -c_0^* + x_1 + x_2 + \cdots + x_T + c_T^* = \sum_{t=0}^{T} x_t + c_T^*$$

so that

$$\bar{\rho}(C) = \frac{x_0 + x_1 + \cdots + x_T + c_T^*}{C}.$$

The net cash flow of the market investment (inclusive of the terminal balance) is then $\sum_1^T x_t + c_T^*$. Note that the latter is equal to rC^*, that is, the net cash flow of the market investment coincides with the total market return. But this implies

$$r = \frac{x_0 + x_1 + \cdots + x_T + c_T^*}{C^*}$$

which is just equal to $\bar{\rho}(C^*)$, the CCOC associated with the total market capital C^*. This result gives us the opportunity of introducing the AROI associated with the market capital.

5. Using the market rate as the hurdle rate

It might be desirable for the analyst to consider the market rate as the hurdle rate. One just needs choose a capital stream belonging to a well-specified investment class, as the following proposition shows.

Proposition 5.1. *(Market rate as CCOC).* Consider the class of those capital streams $\boldsymbol{c}$ such that $C = \sum_0^{T-1} c_t^* = C^*$. We call such a class the *market-investment* class. The market rate is the CCOC associated with the market-investment class.

Proof: by definition of CCOC, $\bar{\rho}(C^*) = rC^*/C^* = r$■

Proposition 5.1 says that the market rate is indeed a correct cost of capital, as long as any capital stream belonging to the market-investment class is selected (e.g. $\boldsymbol{c}^* = (c_0^*, c_1^*, \dots, c_{T-1}^*)$): in this case, the aggregate capital invested in the project coincides with the overall market capital C^*, which implies that the correction factor is inactive, being equal to 1. Therefore, a project is worth undertaking if and only if the market-investment AROI exceeds the market rate: $\bar{k}_0(C^*) > r$ (sign is reversed if $C^* < 0$). Figure 2 depicts a positive-NPV project with $C^* > 0$.

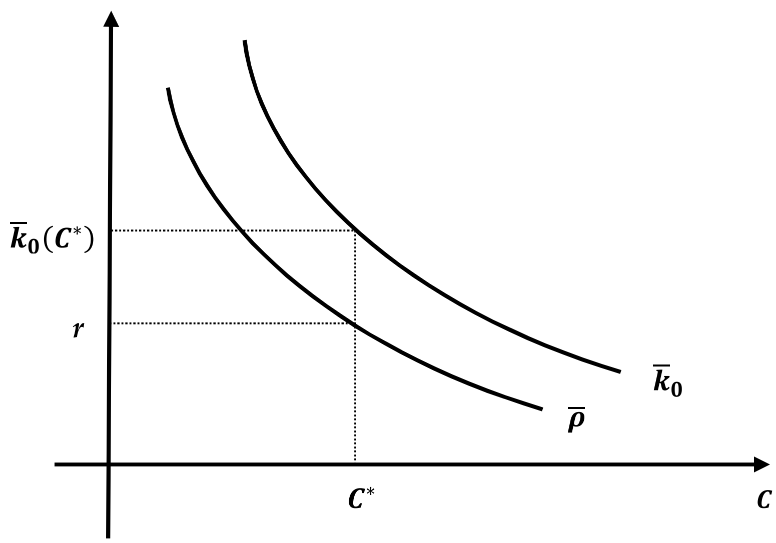


Figure 2. The market rate is the CCOC corresponding to the market investment class.

Remark. The use of the market-investment AROI ($=\bar{k}_0(C^*)$) is a very easy way of solving an accept/reject decision problem. The only thing to do is to divide the project net cash flow by the market capital C^*. The direct comparison with the market rate provides the answer. The economic intuition of this case is as follows. Project $\boldsymbol{x}$ may actually be considered exactly identical to market investment $\boldsymbol{x}^*$, barring the fact that the terminal balance of $\boldsymbol{x}$ is zero. So, the two alternatives share the same cash flows and the same outstanding capitals, which means that the aggregate capitals of the two projects is the same: $C=C^*$. The net cash flow is $\sum_0^T x_t$, and $\sum_0^T x_t/C^*$ is then the return per unit of capital. That very capital C^* could be alternatively invested in the market to earn r per unit of capital. By comparing the two rates the accept/reject decision problem is correctly solved.

6. The internal rate of return as a particular case of AROI

Let k be a real-valued IRR of project $\boldsymbol{x}$; let $c_t(k)=c_{t-1}(k)(1+k)-x_t$ be the IRR-implied outstanding capital. Denoting with $H \coloneqq \sum_0^{T-1} c_t(k)$ the overall capital invested, we consider the IRR-implied investment class (i.e., the set of those capital streams $\boldsymbol{c}$ such that $\sum_0^{T-1} c_t = H$).

Proposition 6.1. (*IRR as AROI*). A (real-valued) IRR is a particular case of AROI generated by an IRR-implied investment class.

Proof: the IRR-implied capital stream $\boldsymbol{c}(k)=(c_0, c_1(k), c_2(k), \dots, c_{T-1}(k))$ is only one among infinitely many capital streams belonging to the same class. Given that H (not $\boldsymbol{c}$) is uniquely associated with $\bar{k}_0$, any capital stream equivalent to $\boldsymbol{c}(k)$ (i.e., belonging to the IRR-implied investment class) gives rise to the same AROI. That is, for any such capital stream,

$$\bar{k}_0(H)=\frac{\sum_1^T k_t c_{t-1}}{H}=\frac{\sum_1^T k c_{t-1}(k)}{H}=k. \blacksquare$$

Therefore, by Theorem 3.1 and Proposition 6.1, a project is worth undertaking if and only if an IRR is greater (smaller if $H<0$) than the corresponding CCOC:

$$\bar{k}_0(H)>\bar{\rho}(H)=r\cdot\frac{C^*}{H}.$$

As seen, any investment class contains infinite elements, so infinite capital streams exist which give rise to that IRR as the AROI of the class, and the internal return vector $\boldsymbol{k}=$

$(k, k, \ldots, k)$ is only one among infinitely many equivalent ones. This means that the IRR is not a mere constant rate applied to the beginning-of-period capital, but a mean of (generally non-constant) period rates weighed by capitals whose algebraic sum is H; the latter is the aggregate capital which is implicitly selected by solving the IRR equation. Therefore, to solve the IRR equation is actually only a convoluted way of choosing a particular investment class and hence a particular AROI (see example in Appendix).

Remark. Proposition 6.1 solves a long-standing problem regarding the relation between accounting rates of return and internal rate of return: "it is widely presumed in the accounting and economic literatures that, for the most part in practice, ARRs [accounting rates of return] are artifacts without economic significance." (Peasnell, 1982a, p. 368). The limitations of using ARRs in place of the IRR has been the focus of several decades of academic research, both theoretical and empirical (see also Harcourt, 1965; Solomon, 1966; Livingstone and Salamon, 1970; Kay, 1976; Fisher and McGowan, 1983. See also Feenstra and Wang's, 2000, review and references therein) and it is widely believed that comparing the ARR with the cost of capital is "clearly like comparing apples with oranges" (Rappaport, 1986, p. 31). Also, a straightforward relation between accounting rates of return and IRR would be welcome:

> Is the average ARP [accounting rate of profit] equal to the IRR in a more obvious sense of average? The general answer is no: it will not normally be true that
>
> $$\bar{a} = \frac{\int_0^\infty a(t)v(t)dt}{\int_0^\infty v(t)} = r$$
>
> where $\bar{a}$ is the *natural definition* of an average ARP. (Kay, 1976, pp. 451, italics supplied).[6]

Proposition 6.1 just shows that, if the capital selected is the book value of the project, then the average accounting rate of return is an AROI corresponding to the capital stream implied by the accounting depreciation policy. As such, it correctly captures the project's economic profitability (contrary to what is believed in the accounting literature). Proposition 6.1 also shows that the general answer to Kay's question is: the IRR is a weighted average of those accounting rates of return that would be generated by an accounting depreciation policy such that the resulting capital stream belongs to the IRR-implied investment class. It is worth stressing that there are infinitely many depreciation schedules such that the

[6] The symbols $v(t)$ in Kay (1976) denotes book value.

sequence of accounting rates generates the IRR: as shown in the proposition, the sequence $\big(c_0, c_1(k), c_2(k), \dots, c_{T-1}(k)\big)$ is sufficient but by no means necessary for producing the IRR; it suffices that the accounting rates of return are obtained from a depreciation schedule where the book values add up to H (see example in the Appendix).

Remark. The IRR is either a particular case of AIRR or a particular case of AROI. What is the difference? If an IRR is seen as a particular case of AIRR, its financial nature may change under changes in the market rate: although the IRR itself does not change, $PV(\boldsymbol{c}(k) \mid r)$ may change (in general) if the market rate changes, so the sign of $PV(\boldsymbol{c}(k) \mid r)$ may change as well, which implies that a project may turn from investment to borrowing (or vice versa). Therefore, an IRR of a project might be a rate of return at a determined point in time, but might abruptly turn to a rate of cost whenever the market rate changes. In contrast, an IRR, seen as a particular case of AROI, keeps its financial nature unaltered, given that it is associated with C, which is not a function of the market rate. Therefore, a genuine internality of the IRR is guaranteed only under the AROI approach.

For example, consider project $\boldsymbol{x} = (-4,\ 3,\ 2.25,\ 1.5,\ 0.75,\ 0, -0.75, -1.5, -2.25)$, previously considered in Hazen (2003) and in Magni (2010a). An IRR of the project is $k = 26.31\%$. Assume the AIRR approach is employed. If the market rate is, say, $r = 11\%$, the project is a net investment, since $PV(\boldsymbol{c}(k)\,|11\%) = 0.144 > 0$: the IRR is then a rate of return. If the market rate decreases to $r = 10\%$, the project becomes a net borrowing, since $PV(\boldsymbol{c}(k)\,|10\%) = -0.113 < 0$, which implies that the IRR turns to a rate of cost. On the contrary, if that IRR is seen as a particular case of AROI, its nature is univocal: it is easy to check that $H = -3.8$, and the project may be seen as a borrowing and the IRR (i.e., the AROI associated to H) is unambiguously a rate of cost, no matter what the market rate is.

Figure 3 illustrates a case of multiple IRRs for a positive-NPV project: denoting with k^1 and k^2 two IRRs, and with H_1 and H_2 the overall capital of the two IRR-determined investment classes, it is immaterial which one of the IRR is selected for measuring economic profitability: each IRR, in association with the corresponding COCC, provides a correct solution to the accept/reject decision problem.

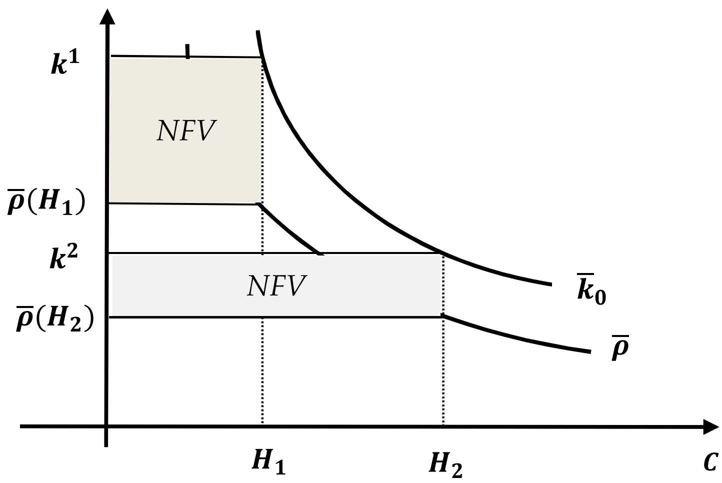


Figure 3. Example of multiple IRRs: each IRR is contrasted with the corresponding CCOC, providing the same answer.

EXAMPLE. Consider project $\boldsymbol{x} = (-9,\ 60, -110,\ 60)$ whose real-valued IRRs are $k^1 = 6.75\%$, $k^2 = 53.76\%$, $k^3 = 306.15\%$. Assuming a market rate equal to $r = 10\%$, the NPV is $PV(\boldsymbol{x}\mid 10\%) = -0.28 < 0$. The market capital stream is $c^* = (9, -50.1, 54.9)$ so that $C^* = 13.8$.

The *NCFC* is $\bar{k}_0(9) = \frac{-9+60-110+60}{9} = 11.11\%$, which is smaller than the corresponding CCOC: $\bar{\rho}(9) = 10\%\left(\frac{13.8}{9}\right) = 15.32\%$. The market-investment AROI is $\bar{k}_0(\,13.8) = \frac{-9+60-110+60}{13.8} = 7.25\%$ which is smaller than the market rate. As for the IRRs, it is easy to check that the corresponding CCOCs are 9.31%, 74.14%, 422.19%, respectively. Table 2 collects the capital invested for each case, as well as the AROIs and CCOCs. Note that RRR= AROI −CCOC > 0, which suggests project acceptance (the project is described as an investment in each case). As is easily checked, the product of total capital and RRR, properly discounted, gives back the NPV.

Table 2. ***NCFC*****, market-investment AROI, internal rate of return**

Project $\boldsymbol{x} = (-9, 60, -110, 60)$, $r = 10\%$, NPV = -0.28

Investment class	Capital Invested	Type	AROI	CCOC
Initial-capital	9	investment	(*NCFC*) 11.11%	15.32%
Market-capital	13.8	investment	7.25%	10%
IRR-implied (k^1)	14.81	investment	6.75%	9.31%
IRR-implied (k^2)	1.86	investment	53.76%	74.14%
IRR-implied (k^3)	0.33	investment	306.15%	422.19%

7. Ranking projects

This section shows that the RRR ranking of a project is equivalent to the NPV ranking. We first need the notion of PV-equivalence. Let $PV(\,C\,|r\,) \coloneqq C(1+r)^{-T}$ be the discounted total capital.

Definition. Two or more capital streams are said to be *PV-equivalent* if they have equal $PV(C \mid r)$.[7]

Theorem 7.1. (*Project ranking*). Consider competing projects $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$. The ranking of the projects via their RRRs is equal to the NPV ranking, provided that the capital streams of the projects are chosen so as to be PV-equivalent.

Proof: First of all, we note that there are infinitely many ways to get the same $PV(C|\,r)$ for the n projects at hand, so the requirement of PV-equivalence gives no problem. Let $\boldsymbol{c}_j$ be the capital stream selected for project $\boldsymbol{x}_j$, $j = 1,2, \dots n$ such that $PV(C|\,r)$ is the same for all projects. Let $\bar{k}_{0,j}$ and $\bar{\rho}_j$ be, respectively, the AROI and the CCOC associated with $\boldsymbol{c}_j$. From eq. (11), it follows that $\max\limits_{1\le j\le n} PV(\boldsymbol{x}_j \mid r)$ is equivalent to $\max\limits_{1\le j\le n} (\bar{k}_{0,j} - \bar{\rho}_j)$ if $PV(C|\,r) > 0$ and to $\min\limits_{1\le j\le n} (\bar{k}_{0,j} - \bar{\rho}_j)$ if $PV(C \mid r) < 0$. ■

EXAMPLE. Consider the following projects: $\boldsymbol{x}_1 = (-100, 10, 10, 110)$, $\boldsymbol{x}_2 = (-90, 69, 10, 12, 20)$, $\boldsymbol{x}_3 = (-35,\ 50,\ -18)$, previously illustrated in Magni (2010a), and let $r = 5\%$.

[7] PV-equivalence in this paper is not PV-equivalence as defined in Magni (2010a). In the latter, two or more capital streams are PV-equivalent if they share an equal $PV(\boldsymbol{c} \mid r)$.

Because $PV(\boldsymbol{x}_1|\,5\%) = 13.6$, $PV(\boldsymbol{x}_2|\,5\%) = 11.6$, $PV(\boldsymbol{x}_3|\,5\%) = -3.7$, the NPV ranking is $\boldsymbol{x}_1 \succ \boldsymbol{x}_2 \succ \boldsymbol{x}_3$. The IRR of the first project is 10%, the IRR of the second project is 12.61%, and no real IRR exists for the third project. This means that the IRR ranking of the three projects is not even possible; also, the first two projects are incorrectly ranked, given that the second project's IRR is greater than the first project's IRR. Let us choose three PV-equivalent capital streams. Among the infinite possible choices, we choose, for illustrative purposes,

$$\boldsymbol{c}_1 = (100,\ 10,\ 5.77)$$
$$\boldsymbol{c}_2 = (90,\ 20,\ 11.2,\ 0.36)$$
$$\boldsymbol{c}_3 = (35,\ 75.25)$$

so that $PV(C\mid 5\%) = 100$. One easily finds

$$\bar{k}_{0,1} = 25.91\% \qquad \bar{\rho}_1 = 12.3\% \implies \bar{k}_{0,1} - \bar{\rho}_1 = 13.62\%$$

$$\bar{k}_{0,2} = 21\% \qquad \bar{\rho}_2 = 6.89\% \implies \bar{k}_{0,2} - \bar{\rho}_2 = 11.6\%$$

$$\bar{k}_{0,3} = -2.72\% \qquad \bar{\rho}_3 = 0.99\% \implies \bar{k}_{0,3} - \bar{\rho}_3 = -3.71\%.$$

The positive RRRs of the first two projects suggest acceptability, whereas the negative RRR of the third project suggests rejection. The RRR ranking is $\boldsymbol{x}_1 \succ \boldsymbol{x}_2 \succ \boldsymbol{x}_3$, equal to the NPV ranking.

The choice of two mutually exclusive investments may be considered a particular case of project ranking, where $n = 2$. However, in this case, an even simpler method can be used. Let $\boldsymbol{x}_i$, $\boldsymbol{x}_j$ be mutually exclusive projects and let $\boldsymbol{x}_{i-j} := \boldsymbol{x}_i - \boldsymbol{x}_j$. The latter is the project which collects the incremental cash flows of $\boldsymbol{x}_i$ over $\boldsymbol{x}_j$. Now, project $\boldsymbol{x}_i$ is preferable to project $\boldsymbol{x}_j$ if and only if $PV(\boldsymbol{x}_i\mid r) > PV(\boldsymbol{x}_j\mid r)$. Due to NPV additivity, this means $PV(\boldsymbol{x}_{i-j}\mid r) > 0$. Therefore, $\boldsymbol{x}_i$ is preferable to $\boldsymbol{x}_j$ if and only if $\boldsymbol{x}_{i-j}$ is acceptable. Letting $\boldsymbol{c}_{i-j} := \boldsymbol{c}_i - \boldsymbol{c}_j$ and $C_{i-j} := \sum_1^T c_t^i - \sum_1^T c_t^j$, we may apply Theorem 3.1 to project $\boldsymbol{x}_{i-j}$ to get the following result.

Proposition 7.1. (*Mutually exclusive investments*). Let $\boldsymbol{x}_i$, $\boldsymbol{x}_j$ be two mutually exclusive alternatives. For any capital stream $\boldsymbol{c}_{i-j} \in \mathbb{R}^T$,

(i) if $C_{i-j} > 0$, $\boldsymbol{x}_i$ is preferable to $\boldsymbol{x}_j$ if and only if $\bar{k}_{0,i-j} > \bar{\rho}_{i-j}$

(ii) if $C_{i-j} < 0$, $\boldsymbol{x}_i$ is preferable to $\boldsymbol{x}_j$ if and only if $\bar{k}_{0,i-j} < \bar{\rho}_{i-j}$

(iii) neither $\boldsymbol{x}_i$ nor $\boldsymbol{x}_j$ is preferable if and only if $\bar{k}_{0,i-j} = \bar{\rho}_{i-j}$.

Proposition 7.1 enables one not to worry about PV-equivalence: the analyst only needs freely fix the capital streams for the two projects[8] and apply the acceptability criterion to the incremental cash flow stream. For example, consider again $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The incremental cash flow stream is $\boldsymbol{x}_{1-2} = (-10, -59, 0, 98, -20)$. The market capital stream is, in this case, $\boldsymbol{c}^* = (10, 69.5, 72.98, -21.38)$ so that the total capital invested is $C_{i-j} = C^* = 131.1$. Using, for simplicity, the market-investment AROI, one gets $\bar{k}_{0,1-2} = \frac{-10-59+98-20}{131.1} = 6.87\% > 5\%$, so project $\boldsymbol{x}_1$ is preferable to project $\boldsymbol{x}_2$. This is confirmed by the NPV of the incremental cash flow stream: $PV(\boldsymbol{x}_{1-2} | \, 5\%) = 131.1 \cdot (6.87\% - 5\%) \cdot (1.05)^{-4} = 2$, which is just equal to $PV(\boldsymbol{x}_1 | \, 5\%) - PV(\boldsymbol{x}_2 | \, 5\%) = 13.6 - 11.6 = 2$.

The procedure may be generalized, and the choice of the preferred project out of n projects may be obtained by a pairwise comparison of projects with the incremental-cash-flow method just described.

8. AROI and AIRR contrasted and the decomposition of NPV

Both the Average Internal Rate of Return and the Aggregate Return On Investment solve the long-standing problem of the IRR. Both are real-valued, and a unique rate of return corresponds to a fixed capital stream. Both are founded on the idea that a project is either an investment or a borrowing depending on how the decision maker describes it, and that any project may be described by the following triplet:

(capital invested, rate of return, cost of capital).

However, the triplet generated by the AIRR depends on the market rate. In particular: the capital invested is the present value of the outstanding capitals ($PV(\boldsymbol{c}, r)$); the rate of return is the mean of period return rates weighted by the present values of the outstanding capitals ($\bar{k}_r$); and the cost of capital is the market rate itself (r). Therefore, the market rate appears in each one of the three variables. By contrast, in the AROI criterion, the capital invested is defined internally, for it is obtained as the algebraic sum of the capitals invested in each period (C); likewise, the rate of return is defined internally as the net cash flow divided by the capital invested ($\bar{k}_0 := \sum_0^T x_t \, / C$) or, equivalently, as the mean of period return rates weighted by the (plain vanilla) outstanding capitals ($\bar{k}_0 := \sum_1^T k_{t-1} c_{t-1} \, / C$); the cost of capital

[8] The analyst does not even need fix the capital streams, if the *NCFC* or the market-investment AROI is employed.

is the only variable affected by the market rate ($\bar{\rho} = rC^*/C$). As a result, a shock in the market may abruptly change the AIRR and even turn the type of the project from borrowing to investment or vice versa (as long as the capital stream is kept fixed). This situation may not occur with the AROI approach, for the exogenous factors are imputed to the cost of capital. A striking implication of this fact is that while an IRR may be viewed as a particular case of both AIRR and AROI, the equivalence is not complete. If an IRR is seen as a special case of AIRR, it is associated with a capital stream which depends on the market rate. This implies that a change in the market rate may turn the nature of the IRR from rate of return to rate of cost or vice versa. In other words, the IRR itself is internal, but the nature of the project (and therefore the nature of the IRR) is not internal. By contrast, an IRR which is drawn from the AROI is either a rate of return or a rate of cost, regardless of the market rate. This means that, if one makes use of an IRR as a project rate of return, its interpretation in the AROI approach enjoys desirable properties of "complete internality".

The triplets supplied by the two indexes lead to the NPV, but the discounting process operates in a rather different way, for the order of sum and discount is reversed: in the AIRR approach, outstanding capitals are discounted first, and then summed; in the AROI approach, outstanding capitals are first summed, and then discounted.

A beneficial characteristic of the AROI is that it is conceptually intuitive, being obtained as cash inflows minus cash outflows divided by capital invested. Such a ratio is rather natural even for people unacquainted with financial reasoning and is computationally very simple. On the other side, the notion of comprehensive cost of capital is a sophisticated one: the hurdle rate is not the market rate, as in the AIRR criterion, but a multiple of it which takes into account the capital of the market investment which the investor foregoes. True as it is, the use of the market-capital AROI in association with the market rate is a very easy way of solving an accept/reject decision problem: one just divides net cash flow by the total market capital and contrast it with the very market rate.

As for project ranking, it is worth noting that while maximization of the AIRR suffices for providing the correct ranking, maximization of the AROI does not provide correct answers: one has to necessarily resort to maximization of the RRR ($= \bar{k}_0 - \bar{\rho}$). From this point of view, the AIRR seems more intuitive, because the same hurdle rate is used for all projects. However, project ranking, as well as choices between two mutually exclusive alternatives, may be easily coped with via application of the acceptability criterion (Theorem 3.1) to the incremental cash flow stream of one project over the other.

While the IRR is a particular case of both AIRR and AROI, one may reverse the perspective and interpret AIRR and AROI as (one-period) IRRs. As for the former, suppose a decision maker has the opportunity of investing in a one-period project $\boldsymbol{y} = (y_0, y_1)$, with $y_0 = -PV(\boldsymbol{c} \mid r)$ and $y_1 = PV(\boldsymbol{c} \mid r)\big(1 + \bar{k}_r\big)$. The NPV of $\boldsymbol{y}$ is

$$PV(\boldsymbol{y} \mid r) = -PV(\boldsymbol{c} \mid r) + \frac{PV(\boldsymbol{c} \mid r)\big(1 + \bar{k}_r\big)}{1+r} = \frac{PV(\boldsymbol{c} \mid r)\big(\bar{k}_r - r\big)}{1+r} \tag{15}$$

which is just $PV(\boldsymbol{x} \mid r)$. So doing, project $\boldsymbol{x}$ is transformed into an economically equivalent one-period project. The IRR of $\boldsymbol{y}$ is $\bar{k}_r$, for $PV_T\big(\boldsymbol{y} \mid \bar{k}_r\big) = 0$.

As for the AROI, project $\boldsymbol{x}$ may be interpreted as a one-period project $\boldsymbol{z} = (z_0, z_1)$, with $z_0 = -C$ and $z_1 = C\big(1 + \bar{k}_0\big)$, fully financed at the interest rate $\bar{\rho}$. Thus, the cash flow stream of the financing is $\boldsymbol{w} = (w_0, w_1)$, with $w_0 = C$ and $w_1 = -C(1 + \bar{\rho})$, and the NFV of the levered project is

$$PV_T(\boldsymbol{z} + \boldsymbol{w} \mid r) = PV_T(\boldsymbol{z} \mid r) + PV_T(\boldsymbol{w} \mid r) = C\big(\bar{k}_0 - r\big) + C(r - \bar{\rho}). \tag{16}$$

The IRR of $\boldsymbol{z}$ is $\bar{k}_0$, for $PV_T\big(\boldsymbol{z} \mid \bar{k}_0\big) = 0$, so the AROI is the IRR of the project *unlevered*. Also, $PV_T(\boldsymbol{z} + \boldsymbol{w} \mid r) = PV_T(\boldsymbol{x} \mid r)$, that is, the NFV of the levered project is just the NFV of $\boldsymbol{x}$**.** It is worth noting that in (16) the NFV is split into two components; reminding that $C(r - \bar{\rho}) = r(C - C^*)$, the first component is the contribution of the excess return rate earned on the invested capital; the second component reflects the contribution of the excess capital invested in $\boldsymbol{x}$. We label the two shares (excess) *rate margin* and (excess) *capital margin*

$$PV_T(\boldsymbol{z} + \boldsymbol{w} \mid r) = PV_T(\boldsymbol{x} \mid r) = \overbrace{C\big(\bar{k}_0 - r\big)}^{\text{rate margin}} + \overbrace{r(C - C^*)}^{\text{capital margin}} \tag{17}$$

In Figure 4, the two components are graphically represented by rectangles, under the assumption $\bar{k}_0 > r$ (positive rate margin). The first case is $C^* < C$, so both rate and capital margins are positive, and the NFV is the sum of the areas of the two rectangles. In the second case, where $C^* > C$, the capital margin is negative so that the NFV is determined by the difference between the areas of the two rectangles. In the third case, $C^* = C$, so the capital margin is nullified (the capital invested in the project coincides with the capital invested in the market investment). (An alternative yet equivalent representation is supplied in Figure 5).

Remark. The AROI approach, as well as the AIRR approach, may be employed with variable market rates. One just has to define CCOC in a more general way: $\bar{\rho} := \frac{\sum_1^T r_t c_{t-1}^*}{C}$, where r_t is the market rate holding in the interval $[t-1, t]$. For example, consider again project $\boldsymbol{x} = (-9,\ 60, -110,\ 60)$ and assume the market rates are $r_1 = 1\%$, $r_2 = 2\%$, $r_3 = 3\%$. The NPV is $-9 + 60(1.01)^{-1} - 110[(1.01)(1.02)]^{-1} + 60[(1.01)(1.02)(1.03)]^{-1} = 0.18$. The market-investment class is $\boldsymbol{c}^* = (9, -50.9, 58.1)$ so that $C^* = 9 - 50.9 + 58.1 = 16.2$. Choosing $C = C^*$, the AROI is $\bar{k}_0 = \frac{-9+60-110+60}{16.2} = 6.19\%$ and the CCOC is just a weighted average of market rates: $\bar{\rho} = \frac{1\%(9)+2\%(-50.9)+3\%(58.1)}{9-50.9+58.1} = 5.04\%$. The project is worth undertaking, for it is an investment ($C = 16.2 > 0$) and $6.19\% > 5.04\%$. The NPV is found back from the RRR: $16.2 \cdot (6.19\% - 5.04\%) \cdot [(1.01)(1.02)(1.03)]^{-1} = 0.18$.

AIRR and AROI are then (different but) companion criteria. The preference of either approach by the analysts should not be made once for all but case by case, depending on the needs, the constraints, and the pieces of information required in the economic analysis. For example, if internality of the return rate and/or unambiguity of the type of the project is a major issue for the analyst, then the AROI criterion will be employed. If the preferred notion of capital invested is "outstanding capital", the AROI criterion will be used; if one prefers the notion of capital invested as "*present value* of the outstanding capital", the AIRR approach will be employed. If the market rate as a hurdle rate is deemed mandatory, then the AIRR approach will be adopted or, alternatively, the market-investment AROI. If a usual IRR is required, then, it is perhaps preferable to consider it as a particular case of AROI, so that the rate is univocally either a rate of return or a rate of cost. If a cash-flow measure is required, then the AROI should be used. In a project ranking process, if ranking of rates of return is regarded as more intuitive than ranking of RRRs, the use of AIRR is mandatory, as well as in the case where the analysts is willing to deal with one single hurdle rate for all projects. Finally, the decomposition of the NPV (NFV) obtained in the AROI approach may provide further information on rates and capitals as the determinants of value creation. (Table 3 collects the major differences between the two measures).

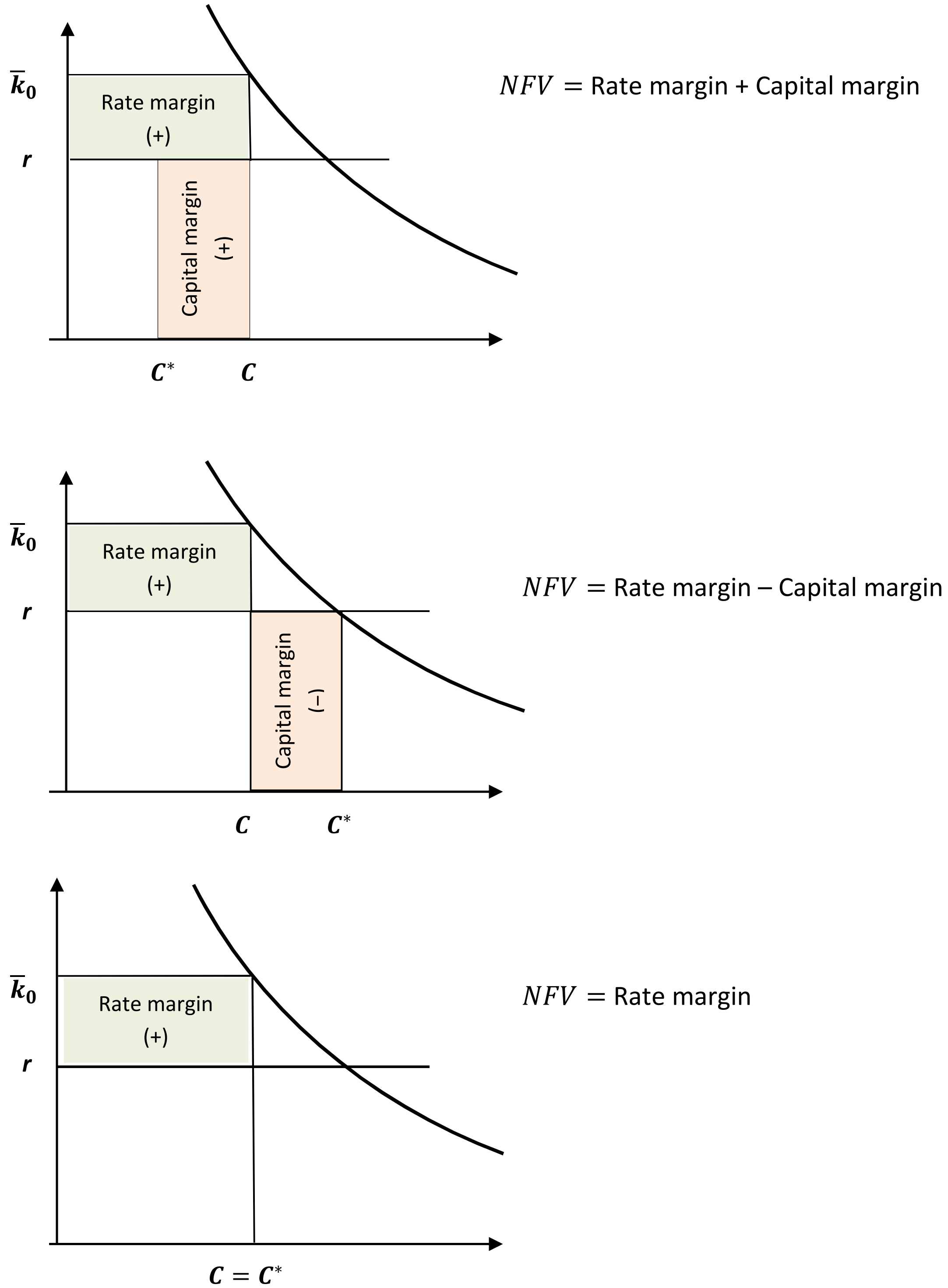
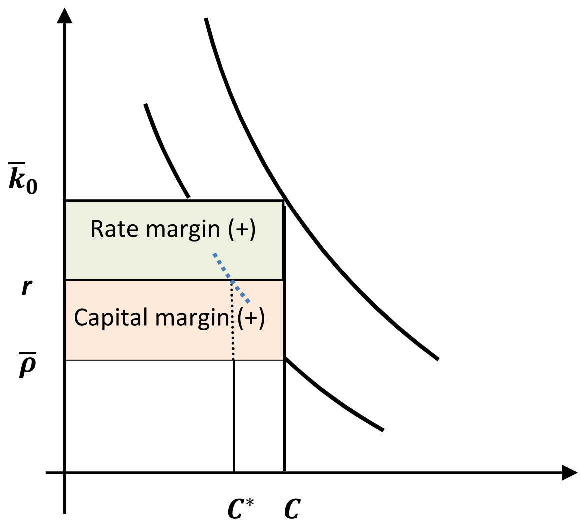
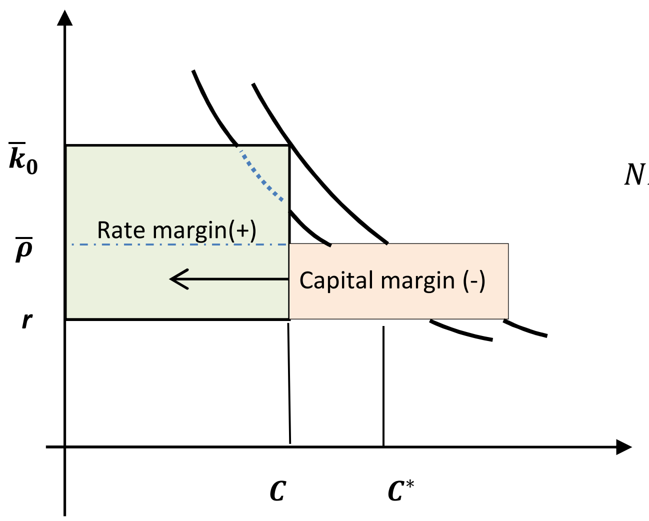


Figure 4. The NFV of a project is decomposed into rate margin and capital margin. From top to bottom: $0 < C^* < C$, $0 < C^* < C$, and $0 < C^* = C$.



NFV = Rate margin + Capital margin



NFV = Rate margin − Capital margin

Figure 5. Alternative representation of the decomposition of the NFV, under the assumptions $0 < C^* < C$ and $0 < C < C^*$ respectively (the case $0 < C = C^*$ is identical to the third graph in Figure 4).

Table 3. AROI and AIRR contrasted

	AROI	AIRR
Definition (return measure)	$\bar{k}_0 := \frac{\sum_1^T R_t}{C}$ $= \frac{\sum_1^T k_t c_{t-1}}{\sum_1^T c_{t-1}}$	$\bar{k}_r := \frac{\sum_1^T R_t(1+r)^{-(t-1)}}{PV(\boldsymbol{c},r)}$ $= \frac{\sum_1^T k_t c_{t-1}(1+r)^{-(t-1)}}{\sum_1^T c_{t-1}(1+r)^{-(t-1)}}$
Definition (cash-flow measure)	$\bar{k}_0 := \frac{\sum_0^T x_t}{C}$	—
dependence on the market rate	NO	YES
Definition of investment	$C > 0$	$PV(\boldsymbol{c},r) > 0$
dependence on the market rate	NO	YES
Hurdle rate	$\bar{\rho} := r \cdot \left(\frac{C^*}{C}\right)$	r
dependence on the market rate	YES	YES
Residual rate of return	$\bar{k}_0 - \bar{\rho}$	$\bar{k}_r - r$
Criterion of project acceptability*	$\bar{k}_0 > \bar{\rho}$	$\bar{k}_r > r$
Criterion of project ranking	$\max\,(\bar{k}_0 - \bar{\rho})$	$\max\,(\bar{k}_r - r) = \max \bar{k}_r$
Hurdle rate with variable market rates	$\bar{\rho} := \frac{\sum_1^T r_t c_{t-1}^*}{C}$	$\bar{r} := \frac{\sum_1^T r_t c_{t-1}}{C}$
Net Present Value	$C(\bar{k}_0 - \bar{\rho})(1+r)^{-T}$	$PV(\boldsymbol{c},r)(\bar{k}_r - r)(1+r)^{-1}$
Rate margin	$C(\bar{k}_0 - r)(1+r)^{-T}$	
Capital margin	$r(C - C^*)(1+r)^{-T}$	

* The criterion includes the incremental-cash-flow method for choosing between two mutually exclusive investments.

Concluding remarks

In a recent paper, Magni (2010a) proposes a complete solution to the long-standing problem of the Internal Rate of Return (IRR). The Average Internal Rate of Return (AIRR), a weighted average of period return rates, is capable of correctly solving accept/reject decision problems and ranking a bundle of projects. This paper investigates the possibility of employing an intuitive cash-flow measure for making investment decisions: cash inflows minus cash outflows divided by capital invested. This ratio, which we call Aggregate Return On Investment (AROI), coincides with a weighted average of the project's period return rates. This cash-flow measure is independent of the market rate, so it is a purely internal rate. The AROI successfully copes with accept/reject decision problems and with project ranking, as long as the hurdle rate is a comprehensive cost of capital, which takes into consideration that the investor undertaking the project foregoes the opportunity of replicating the project in the market. As a beneficial byproduct, the AROI approach enables the analyst to compellingly decompose the NPV (NFV) into two shares: one is generated by the difference between rates of return (project rate versus market rate), the other one is generated by the difference between the capitals invested (project capital versus market capital). Such an apportionment sheds light on the determinants of value as well as their relative importance (rate versus capital).

The AROI enriches the toolkit of the engineering economist: AROI, AIRR and NPV may be interchangeably used for coping with investment decisions; and the classical IRR, when it is real-valued, is a particular case of both AROI and AIRR. The three approaches are financially equivalent, and whether one will make use of either criterion is only a matter of needs, constraints, practical issues, subjective tastes, purpose of the analysis, or institutional commitment.

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Appendix

The IRR is a weighted average of (generally non-constant) period rates. Table 4 collects some capital streams for project $\boldsymbol{x} = (-900, 800, 100, 100, 91)$ and compute the corresponding sequences of period rates. The IRR is 14%, and while the constant sequence (14%, 14%,…,14%) does obviously produce 14% as a mean, such a sequence is not necessary to generate the IRR: it suffices that the aggregate capital is equal to 1363.6. If one chooses the IRR as the relevant rate of return for this project, then one is describing the project as an investment of 1363.6 dollars: the return on each dollar is 0.14 dollars.

Table 4. IRR as a weighted average of period rates

Time	0	1	2	3	4	AROI (%)
Cash Flows	−900	800	100	100	91	
Capital stream 1	900	226.1	157.7	79.8	0	
Period rates (%)		14	14	14	14	14
Capital stream 2	900	200	100	163.6	0	
Period rates (%)		11.11	0	163.6	−44.38	14
Capital stream 3	900	300	123.6	40	0	
Period rates (%)		22.22	−25.47	13.27	127.5	14
Capital stream 4	900	0	93.3	370.3	0	
Period rates (%)		−11.11	undefined*	404.07	−75.43	14
Capital stream 5	900	190	157	116.6	0	
Period rates (%)		10	35.26	37.96	−21.96	14
Capital stream (generic)	900	c_1	c_2	$1363.6 - c_1 - c_2$	0	
Period rates (%)		k_1	k_2	k_3	k_4	14

*When the capital is zero, the period rate is undefined, but the AROI is nonetheless defined setting $C = 1363.6$ in eq. (13).

Biographical sketch. Carlo Alberto Magni, Associate Professor, is affiliated with the Department of Economics of the University of Modena and Reggio Emilia, and with CEFIN – Center for Research in Banking and Finance, and is a member of the European Accounting Association. Graduated in Economics and Business with honors, he received his Ph.D. in Mathematics applied to economic problems from the University of Trieste. He holds a Master in Business Administration from the University of Turin. His research interests include decision analysis, finance and accounting, economic methodology. He is currently involved in the following topics: accounting and economic rates of return, residual income, net present value and CAPM, investment evaluations and arbitrage, accounting-based valuation, performance measurement, economic modeling, bounded rationality and heuristics, fuzzy expert systems, sensitivity analysis. He has published in several international journals. His teaching activities include calculus, financial mathematics and capital budgeting, mathematics for economics and finance.